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LETTER TO THE EDITOR

Calculation of crossover exponent from Heisenberg to Ising behaviour using the fourth-order ε expansion

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Abstract. We extend the ε expansion for the crossover exponent ϕ from Heisenberg to Ising behaviour to fourth order in $\varepsilon = 4-d$. Resummation of this expansion incorporating the high-order behaviour gives numerical estimates in three dimensions which are in agreement with high-temperature series expansions and experimental results.

In this letter we consider the application of the field theoretic method to the evaluation of the crossover exponent ϕ from Heisenberg to Ising behaviour. We construct a fourth-order ε expansion for ϕ and then resum incorporating the high-order behaviour.

We begin by giving a precise description of ϕ and then explain how it can be related to an anomalous dimension of the n-component Landau–Ginzburg model. Consider a Heisenberg system with a weak spin anisotropy; if we decrease the temperature towards the critical point, the behaviour of the system changes in the neighbourhood of some temperature T^x from the isotropic to the anisotropic behaviour. If

$$\Delta T = T^x - T_c(g) \tag{1}$$

where T_c is the critical temperature and g the coupling constant of the anisotropy then

$$\Delta T \sim g^{1/\phi} \tag{2}$$

defines the crossover exponent ϕ. It can be shown (Pfeuty *et al* 1974) that this gives an extended scaling hypothesis for the susceptibility

$$\chi(T, g) \sim At^{-\gamma} \times (Bg/t^{\phi}) \tag{3}$$

where $t = (T-T_c)/T_c$, T_c is the critical temperature and γ the susceptibility exponent of the isotropic model. Consider the n-component Landau–Ginzburg Hamiltonian with a small anisotropy term

$$\mathscr{H} = \int \mathrm{d}^d x \left(\tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}m^2\varphi^2 + \frac{u_0}{4!}(\varphi^2)^2 + \tfrac{1}{2}g d_{ab}\varphi_a\varphi_b \right) \tag{4}$$

where d_{ab} is a symmetric traceless tensor.

$$(\nabla\varphi)^2 = \sum_{a=1}^{d}\sum_{i=1}^{n} \nabla_a\varphi_i\nabla_a\varphi_i \qquad \varphi^2 = \sum_{i=1}^{n}\varphi_i\varphi_i.$$

We introduce a renormalised field $\varphi_R = \varphi Z^{-1/2}$, renormalised couplings $u_R\mu^{\varepsilon}$, g_R, and a renormalised mass m_R^2 in the usual way. The Hamiltonian (4) written in terms of these new variables is

$$\mathscr{H} = \int \mathrm{d}^d x \Big(\tfrac{1}{2}(\nabla\varphi_R)^2 + \tfrac{1}{2} m_R^2 \varphi_R^2 + \frac{u_R \mu^\varepsilon}{4!}(\varphi_R^2)^2 + \tfrac{1}{2} g_R d_{ab}\varphi_{Ra}\varphi_{Rb} + \tfrac{1}{2}(Z-1)(\nabla\varphi_R)^2$$
$$+ \tfrac{1}{2} m_R^2 (Z_{\phi^2} - 1)\varphi_R^2 - \mu^\varepsilon \frac{1}{4!}(u_R - u_0' Z^2)(\varphi^2)^2 + \tfrac{1}{2} d_{ab}\varphi_{Ra}\varphi_{Rb} g_R [Z_{\varphi\varphi} - 1] \Big) \tag{5}$$

where $Z_{\phi^2} = m^2/m_R^2$ is the mass renormalisation, $u_0' = \mu^{-\varepsilon} u_0$ and $Z_{\varphi\varphi} = g/g_R$ is the renormalisation associated with the small anisotropy. We define

$$\gamma_\varphi(u_R) = \beta(u_R)\frac{\partial \ln Z}{\partial u_R} \qquad \gamma_{\varphi^2}(u_R) = \beta(u_R)\frac{\partial \ln Z_{\varphi^2}}{\partial u_R} \qquad \gamma_{\varphi\varphi}(u_R) = \beta(u_R)\frac{\partial \ln Z_{\varphi\varphi}}{\partial u_R} \tag{6}$$

where $\beta(u_R)$ is the renormalisation function

$$\beta(u_R) = -\varepsilon \left[\frac{\partial \ln u_0'}{\partial u_R}\right]^{-1} \tag{7}$$

of the isotropic theory. We can show that ϕ is given in terms of the γ's by

$$\phi = \frac{2 - \gamma_\varphi(u^*) + \gamma_{\varphi\varphi}(u^*)}{2 - \gamma_\varphi(u^*) + \gamma_{\varphi^2}(u^*)} \tag{8}$$

where u^* is the fixed point of the isotropic theory. The values of γ_φ, γ_{φ^2} and u^* have already been calculated at fourth order by Vladimirov *et al* (1979). The evaluation of $\gamma_{\varphi\varphi}(u^*)$ to fourth order therefore will allow us to calculate φ. The simplest vertex from which to calculate $Z_{\varphi\varphi}$ is the two-point function with one anisotropic insertion. We denote this by $\Gamma_{abij}^{(2,0,1)}$ where the 0 denotes the absence of φ^2 insertions and i and j label the external legs.

$$\Gamma_{Rabij}^{(2,0,1)} = g_R d_{ab}\delta_{ij} + (Z_{\varphi\varphi} - 1) d_{ab}\delta_{ij} g_R + \Omega_{abij} \tag{9}$$

where Ω is the sum of all one particle irreducible Feynman diagrams with two external legs and one anisotropic insertion. The diagrams contributing at one and two loops are shown in figure 1. Each graph must be proportional to $d_{ab}\delta_{ij}$ by symmetry, so we can define

$$\Gamma_R^{(2,0,1)} = g_R + (Z_{\varphi\varphi} - 1) g_R + \Omega. \tag{10}$$

We can thus calculate $Z_{\varphi\varphi}$ in the minimal subtraction scheme ('t Hooft and Veltman 1972) from the sum of the divergent parts of these diagrams. We use the skeleton

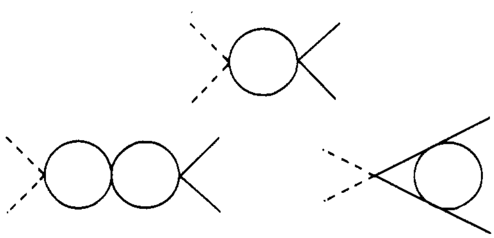

Figure 1. Feynman diagrams which contribute to Ω at one and two loops.

technique which is described in Vladimirov (1979) and also in de Alcantara Bonfim *et al* (1981) to simplify the evaluation of most of the diagrams. For some diagrams, it is easier to use an external momentum q and a massless propagator rather than a massive propagator to control the infrared divergences. We can do this because within a minimal subtraction scheme the total renormalisation required for a given skeleton structure is independent of the mass and momentum. In order to calculate $\gamma_{\varphi\varphi}(u_R)$ we need $\beta(u_R)$ at three loops. We recalculate this in our renormalisation scheme as it is not a universal quantity. The result for $\gamma_{\varphi\varphi}(u_R)$ is

$$\gamma_{\varphi\varphi}(u_R)=\frac{-u_R}{3}+u_R^2\frac{(n+6)}{36}+u_R^3\left(\frac{2n^2-47n-230}{27\times 32}\right)$$
$$+u_R^4\left(\frac{\zeta(3)}{4\times 3^5}(-\tfrac{3}{8}n^3+51)+\frac{\zeta(4)(5n+22)}{3^3\times 8}\right.$$
$$\left.+\frac{1}{3^5\times 64}(3n^3+65n^2+2580n+8140)\right) \tag{11}$$

where $\zeta(p)$ is the pth order Riemann zeta function. We substitute $u_R=u^*$ using u^* to $O(\varepsilon^4)$ from Vladimirov *et al* (1979) to obtain

$$\gamma_{\varphi\varphi}(u^*)=\frac{-2\varepsilon}{n+8}+\frac{\varepsilon^2}{(n+8)^3}(n^2-4n-36)+\frac{\varepsilon^3}{(n+8)^5}[24(n+8)(5n+22)\zeta(3)$$
$$+\tfrac{1}{2}n^4+\tfrac{45}{2}n^3+95n^2-72n-784]+\varepsilon^4\left(\frac{-80\zeta(5)(2n^2+55n+186)}{(n+8)^5}\right.$$
$$+\frac{18\zeta(4)(5n+22)}{(n+8)^4}+\frac{\zeta(3)}{(n+8)^7}(-\tfrac{1}{2}n^6-12n^5-468n^4-5044n^3$$
$$-11\,376n^2+40\,224n+122\,112)+\frac{1}{(n+8)^7}(\tfrac{1}{4}n^6+\tfrac{135}{8}n^5+459n^4$$
$$\left.+3321n^3+10\,941n^2+15\,408n+752)\right)+O(\varepsilon^5). \tag{12}$$

The third-order result is in agreement with that of Yamazaki (1974). We can rewrite equation (8) in a more useful form

$$\phi=(2-\eta+\gamma_{\varphi\varphi}(u^*))\nu \tag{13}$$

since $\nu^{-1}-2+\eta=\gamma_{\varphi^2}(u^*)$ and $\eta=\gamma_\varphi(u^*)$.

ϕ can then be easily evaluated as an ε expansion by substitution of our value of $\gamma_{\varphi\varphi}(u^*)$ and η and ν from Vladimirov *et al* (1979). We find

$$\phi=1+\frac{\varepsilon n}{2(n+8)}+\frac{\varepsilon^2 n(n^2+24n+68)}{4(n+8)^3}$$
$$+\varepsilon^3\left[\frac{-6n(5n+22)\zeta(3)}{(n+8)^4}+\frac{n}{(n+8)^5}\left(\frac{n^4}{8}+6n^3+\frac{197n^2}{2}+434n+682\right)\right]$$
$$+\varepsilon^4\left[\frac{20n(2n^2+55n+186)\zeta(5)}{(n+8)^5}-\frac{9n(5n+22)\zeta(4)}{2(n+8)^4}\right.$$

$$+\frac{\zeta(3)n}{(n+8)^7}(-n^5+5n^4-440n^3-9068n^2-46\,088n-66\,880)$$
$$+\frac{n}{(n+8)^7}\left(\frac{n^6}{16}+\frac{9n^5}{2}+\frac{2085n^4}{16}+\frac{7103n^3}{4}\right.$$
$$\left.+\frac{36\,777n^2}{4}+21\,072n+19\,140\right)\Big]+\mathrm{O}(\varepsilon^5). \qquad (14)$$

We see that as $n\to 0$, $\phi=1$ as expected. In the limit $n\to\infty$, the leading order in n^{-1} expansion is

$$\phi=\frac{1}{1-\frac{1}{2}\varepsilon}-\frac{4\varepsilon}{n}\frac{\sin(\frac{1}{2}\pi\varepsilon)}{(\frac{1}{2}\pi\varepsilon)}\frac{\Gamma(2-\varepsilon)}{(\Gamma(1-\frac{1}{2}\varepsilon))^2}+\mathrm{O}(n^{-2}) \qquad (15)$$

which can be calculated using scaling from the n^{-1} expansions for the critical exponent γ and $\gamma_{\varphi\varphi}(u^*)$ which are derived in Wilson (1973), Wallace (1973) and Wallace and Zia (1975). If we expand (15) in ε to $\mathrm{O}(\varepsilon^4)$ we obtain

$$\phi=1+\tfrac{1}{2}\varepsilon+\tfrac{1}{4}\varepsilon^2+\tfrac{1}{8}\varepsilon^3+\tfrac{1}{16}\varepsilon^4+n^{-1}[-4\varepsilon+\varepsilon^3+\varepsilon^4-\zeta(3)\varepsilon^4]+\mathrm{O}(\varepsilon^5)+\mathrm{O}(n^{-2}) \qquad (16)$$

and this is the same as the expansion to $\mathrm{O}(n^{-1})$ of (14). We used the same methods for evaluating numerical estimates as in de Alcantara Bonfim *et al* (1981) where a more detailed account may be found. The ε-expansion series is asymptotic and has the high-order behaviour

$$(-1)^K K!\,a^K K^b C\varepsilon^K(1+\mathrm{O}(K^{-1})) \qquad (17)$$

for K large enough (Lipatov 1977, Brézin *et al* 1977). The values of a and b for $\gamma_{\varphi\varphi}(u^*)$ are

$$a=\frac{3}{n+8} \qquad b=4+\tfrac{1}{2}n.$$

These are the same as those for $\gamma_{\varphi^2}(u^*)$ which are derived in Brézin *et al* (1977). We can see this in the following way: at high order in perturbation theory, we know from instanton calculations that below the critical dimension the dominant contributions come from the irreducible diagrams (diagrams which have no subdivergences). In a φ^4 theory, there are no irreducible diagrams in either the two-point function or the vertex functions with insertions. All the irreducible diagrams occur in the four-point function and so contribute to the fixed point u^*. When we calculate $\gamma_{\varphi\varphi}(u^*)$ and $\gamma_{\varphi^2}(u^*)$ the behaviour of u^* dominates and so they must have the same values of a and b and further the ratio of the C must be that of the leading orders in u_R i.e. $2/(n+2)$. We can use this ratio as a guide to whether the asymptotic regime has been reached. In particular, we calculate the ratio

$$R_k(n)=\frac{[2/(n+2)]\gamma^k_{\varphi^2}}{\gamma^k_{\varphi\varphi}} \qquad R_k(n)\to 1 \quad \text{as } k\to\infty \qquad (18)$$

where $\gamma^k_{\varphi^2}$ and $\gamma^k_{\varphi\varphi}$ are the coefficients of ε^k in $\gamma_{\varphi^2}(u^*)$ and $\gamma_{\varphi\varphi}(u^*)$ respectively. We find that

$$R_3(2)=0.81 \qquad R_4(2)=1.13$$
$$R_3(3)=1.37 \qquad R_4(3)=0.89$$

and so the fourth-order terms are coming close to the asymptotic behaviour. We are thus able to apply the conformal mapping technique using $a = 3/(n+8)$, $b = 4 + \frac{1}{2}n$ following precisely the same method as described in Vladimirov (1979) and de Alcantara Bonfim *et al* (1981). We find that

$$\gamma_{\varphi\varphi}(u^*) = \begin{cases} -0.207 & n = 2 \\ -0.181 & n = 3 \end{cases} \tag{19}$$

for $\varepsilon = 1$. If instead we calculate using the Padé–Borel method and a [2, 2] approximant we find

$$\gamma_{\varphi\varphi}(u^*) = \begin{cases} -0.207 & n = 2 \\ -0.182 & n = 3 \end{cases} \tag{20}$$

which are only different from the above in the third significant figure. In order to compare our results with other methods we need to evaluate ϕ. We cannot do this directly as the series for ϕ has not reached its asymptotic regime; for $n = 3$ all terms are of the same sign at fourth order in ε. We therefore calculated ϕ from $\gamma_{\varphi\varphi}(u^*)$ using the scaling relation (13). We use the values of η and ν obtained by Le Guillou and Zinn-Justin (1980) from the resummation of the six loop expansion in three dimensions. The result is

$$\phi = \begin{cases} 1.177 & n = 2 \\ 1.259 & n = 3. \end{cases} \tag{21}$$

In table 1 we compare our results with those of the high-temperature series expansions and experiments and notice that they always lie within the error bounds of these results.

Table 1.

ϕ	Results from ε expansion	High-temperature series	Experiment
$n = 2$	1.177	1.175 ± 0.015[1]	1.18 ± 0.05[2]
$n = 3$	1.259	1.25 ± 0.015[1]	1.278 ± 0.076[4]
			1.274 ± 0.045[4]

[1] Pfeuty *et al* (1974).
[2] Domann (1979) for $TbPO_4$.
[3] Basten *et al* (1979a, b, 1980) for $CsMnBr_3.2D_2O$.
[4] Shapira and Oliveira (1978a, b, c) for two different samples of $RbMnF_3$.

In conclusion, the values of ϕ we have found are in agreement with other methods. In view of the fact that our series is so short we have not carried out the more careful kind of resummation analysis used by Le Guillou and Zinn-Justin (1980) which also gives error estimates. It seems reasonable to assume by comparison of the Padé-Borel and conformal mapping techniques that the error at this order is less than 2%.

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